

Invariant manifolds for finite-dimensional non-archimedean dynamical systems

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Abstract

Let M be an analytic manifold modelled on an ultrametric Banach space over a complete ultrametric field \mathbb{K} . Let $f: M \rightarrow M$ be an analytic diffeomorphism and p be a fixed point of f . We discuss invariant manifolds around p , like stable manifolds, centre-stable manifolds and centre manifolds, with an emphasis on results specific to the case that M has finite dimension. The results have applications in the theory of Lie groups over totally disconnected local fields.

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Introduction and statement of main results

Guided by the classical theory of invariant manifolds for time-discrete smooth dynamical systems over the real ground field (cf. [11], [13], [14], [23]), invariant manifolds have recently also been constructed for time-discrete analytic dynamical systems over a complete ultrametric field $(\mathbb{K}, |\cdot|)$ [9]. The invariant manifolds are useful in the theory of Lie groups over local fields, where they allow results to be extended to ground fields of positive characteristic, which previously were available only in characteristic 0 (i.e., for p -adic Lie groups). To enable these Lie theoretic applications, the general theory from [9] is not sufficient, and additional, more specific results concerning ultrametric invariant manifolds are needed. The goal of this article is to provide such complementary results, including simplifications of the theory from [9] for finite-dimensional dynamical systems, which make it applicable in the situations at hand.

As in the real case, hyperbolicity assumptions are essential for a discussion of invariant manifolds. Roughly speaking, a continuous linear self-map

$\alpha: E \rightarrow E$ of an ultrametric Banach space E over \mathbb{K} is called hyperbolic if E admits a decomposition $E = E_s \oplus E_u$ into a stable subspace E_s on which α is contractive and an unstable subspace E_u on which α is expansive. More precisely, α is called *hyperbolic* if it is 1-hyperbolic in the following sense [9]:

Definition. The continuous linear map $\alpha: E \rightarrow E$ is said to be *a-hyperbolic* for $a \in]0, \infty[$ if there exist α -invariant vector subspaces $E_{a,s}$ and $E_{a,u}$ of E such that $E = E_{a,s} \oplus E_{a,u}$, and an ultrametric norm $\|\cdot\|$ on E defining its topology, with properties (a)–(c):

- (a) $\|x + y\| = \max\{\|x\|, \|y\|\}$ for all $x \in E_{a,s}$ and $y \in E_{a,u}$;
- (b) $\alpha_2 := \alpha|_{E_{a,u}}$ is invertible;
- (c) $\|\alpha_1\| < a$ and $\frac{1}{\|\alpha_2^{-1}\|} > a$ holds for the operator norms with respect to $\|\cdot\|$, where $\alpha_1 := \alpha|_{E_{a,s}}$ (and $\frac{1}{0} := \infty$).

Then $E_{a,s}$ is uniquely determined and if α is invertible or E finite-dimensional, then also $E_{a,u}$ is unique (see [9, Remark 6.4] and Remark 1.3 below). If $a = 1$, we also write $E_s := E_{1,s}$ and $E_u := E_{1,u}$.

Similarly, E may have an a -centre-stable subspace $E_{a,cs}$ such that

$$E = E_{a,cs} \oplus E_{a,u},$$

or an a -centre subspace $E_{a,c}$ such that

$$E = E_{a,s} \oplus E_{a,c} \oplus E_{a,u};$$

see Definitions 1.4 and 1.5 for details. We omit the subscript “ a ” if $a = 1$.

It is useful to fix a notation for the set of absolute values of eigenvalues, in the finite-dimensional case.

Definition. Let $\alpha: E \rightarrow E$ be a linear self-map of a finite-dimensional vector space E over a complete ultrametric field $(\mathbb{K}, |\cdot|)$. We use the same symbol, $|\cdot|$, for the unique extension of $|\cdot|$ to an absolute value on an algebraic closure $\overline{\mathbb{K}}$ of \mathbb{K} (see [17, Theorem 16.1]). We write $R(\alpha) \subseteq [0, \infty[$ for the set of all $|\lambda|$ such that $\lambda \in \overline{\mathbb{K}}$ is an eigenvalue of $\alpha \otimes_{\mathbb{K}} \text{id}_{\overline{\mathbb{K}}}$.

The above definition of hyperbolicity is a good basis for theorems, but may be

difficult to verify directly. Fortunately, in the finite-dimensional case, an easier (and more concrete) description of hyperbolicity can be obtained. Also, the existence of centre subspaces and centre-stable subspaces is automatic:

Theorem A. *Let $\alpha: E \rightarrow E$ be a linear self-map of a finite-dimensional vector space E over a complete ultrametric field \mathbb{K} . Then E admits an a -centre-stable subspace and an a -centre subspace, for each $a \in]0, \infty[$. Moreover, α is a -hyperbolic if and only if $a \notin R(\alpha)$.*

Let M be an analytic manifold modelled on an ultrametric Banach space E over \mathbb{K} (as in [3]). An analytic diffeomorphism $\kappa: U \rightarrow V$ from an open set $U \subseteq M$ onto an open set $V \subseteq E$ is called a *chart* for M . An analytic map $f: N \rightarrow M$ between analytic manifolds is called an *immersion* if, for each $x \in N$, the tangent map $T_x(f): T_x(N) \rightarrow T_{f(x)}(M)$ is a homeomorphism onto its image $\text{im } T_x(f)$, and $\text{im } T_x(f)$ is complemented in $T_{f(x)}(M)$ as a topological vector space. If M and N have finite dimension, this simply means that $T_x(f)$ is injective for each $x \in N$. An analytic manifold N is called an *immersed submanifold* of M if $N \subseteq M$ as a set and the inclusion map $\iota: N \rightarrow M$ is an immersion. For $x \in N$, we identify $T_x(N)$ with the vector subspace $\text{im } T_x(\iota)$ of $T_x(M)$.

As before, let M be an analytic manifold modelled on an ultrametric Banach spaces E over a complete ultrametric field $(\mathbb{K}, |\cdot|)$. Let $f: M \rightarrow M$ be an analytic diffeomorphism, and $p \in M$ be a fixed point of f .

Definition. Given $a \in]0, 1]$, we define $W_a^s(f, p) \subseteq M$, the a -stable set around p with respect to f , as the set of all $x \in M$ such that

$$f^n(x) \rightarrow p \text{ as } n \rightarrow \infty \text{ and } a^{-n} \|\kappa(f^n(x))\| \rightarrow 0, \quad (1)$$

for some (and hence every) chart $\kappa: U \rightarrow V \subseteq E$ of M with $p \in U$ such that $\kappa(p) = 0$, and some (and hence every) ultrametric norm $\|\cdot\|$ on E defining its topology.¹

It is clear from the definition that $W_a^s := W_a^s(f, p)$ is stable under f , i.e., $f(W_a^s) = W_a^s$. If the tangent map $T_p(f): T_p(M) \rightarrow T_p(M)$ is a -hyperbolic (which can be checked using Theorem A), then W_a^s is an analytic manifold, the a -stable manifold around p with respect to f :

¹See [9, Remark 6.5] for the independence of the choice of κ and $\|\cdot\|$.

Ultrametric Stable Manifold Theorem (cf. [9, Theorem 1.3]). *Let M be an analytic manifold modelled on an ultrametric Banach space over a complete ultrametric field \mathbb{K} . Let $f: M \rightarrow M$ be an analytic diffeomorphism, $p \in M$ be a point fixed by f , and $a \in]0, 1]$. If the tangent map $\alpha := T_p(f): T_p(M) \rightarrow T_p(M)$ is a -hyperbolic (which is satisfied if M is finite-dimensional and $a \notin R(\alpha)$), then there exists a unique analytic manifold structure on $W_a^s := W_a^s(f, p)$ such that (a)–(c) hold:*

- (a) W_a^s is an immersed submanifold of M ;
- (b) W_a^s is tangent to the a -stable subspace $T_p(M)_{a,s}$ (with respect to $T_p(f)$), i.e., $T_p(W_a^s) = T_p(M)_{a,s}$;
- (c) f restricts to an analytic diffeomorphism $W_a^s \rightarrow W_a^s$.

Moreover, each neighbourhood of p in W_a^s contains an open neighbourhood Ω of p in W_a^s which is a submanifold of M , is f -invariant (i.e., $f(\Omega) \subseteq \Omega$), and such that $W_a^s = \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$.

If $T_p(f)$ is hyperbolic, then W_1^s is simply called the *stable manifold* around p , and denoted W^s .

Now consider the following local situation:

Let M be an analytic manifold modelled on an ultrametric Banach space over a complete ultrametric field \mathbb{K} . Let $M_0 \subseteq M$ be open, $f: M_0 \rightarrow M$ be an analytic mapping, $p \in M_0$ be a fixed point of f , and $a \in]0, 1]$. The following four definitions are taken from [9].

Definition. If $T_p(M)$ has an a -centre-stable subspace $T_p(M)_{a,cs}$ with respect to $T_p(f)$, we call an immersed submanifold $N \subseteq M_0$ an *a -centre-stable manifold* around p with respect to f if (a)–(d) are satisfied:

- (a) $p \in N$;
- (b) N is tangent to $T_p(M)_{a,cs}$ at p , i.e., $T_p(N) = T_p(M)_{a,cs}$;
- (c) $f(N) \subseteq N$; and
- (d) $f|_N: N \rightarrow N$ is analytic.

If $a = 1$, we simply speak of a *centre-stable manifold*.

Definition. If $T_p(f)$ is an automorphism and $T_p(M)$ has a centre subspace $T_p(M)_c$ with respect to $T_p(f)$, we say that an immersed submanifold $N \subseteq M_0$ is a *centre manifold* around p with respect to f if (a), (c) and (d) from the preceding definition hold as well as

(b)' N is tangent to $T_p(M)_c$ at p , i.e., $T_p(N) = T_p(M)_c$.

Definition. In the situation above, assume that $T_p(f)$ is a -hyperbolic. An immersed submanifold $N \subseteq M_0$ is called a *local a -stable manifold* around p with respect to f if (a), (c) and (d) just stated are satisfied as well as

(b)" N is tangent at p to the a -stable subspace $T_p(M)_{a,s}$ with respect to $T_p(f)$, i.e., $T_p(N) = T_p(M)_{a,s}$.

If $a = 1$, we simply speak of a *local stable manifold*.

Definition. In the situation above, assume that $T_p(f)$ is a -hyperbolic. An immersed submanifold $N \subseteq M_0$ is called a *local a -unstable manifold* around p with respect to f if

- (a) $p \in N$;
- (b) N is tangent at p to the a -unstable subspace $T_p(M)_{a,u}$ with respect to $T_p(f)$, i.e., $T_p(N) = T_p(M)_{a,u}$;
- (c) There exists an open neighbourhood U of p in N such that $f(U) \subseteq N$ and $f|_U: U \rightarrow N$ is analytic.

Combining Theorem A with [9, Theorems 1.9, 1.10, 6.6 and 8.3] (which contain further information), we obtain in the finite-dimensional case:

Local Invariant Manifold Theorem. *Let M be a finite-dimensional analytic manifold over a complete ultrametric field \mathbb{K} , $M_0 \subseteq M$ be an open subset, $f: M_0 \rightarrow M$ be an analytic map and $p \in M_0$ a point fixed by f . If $a \in]0, 1]$, then (a)–(c) hold:*

- (a) *There exists an a -centre-stable manifold N around p with respect to f , such that N is a submanifold of M ;*
- (b) *If $\alpha := T_p(f)$ is an automorphism, then there exists an a -centre manifold N around p with respect to f which is a submanifold of M , such that $f(N) = N$;*

- (c) If $a \notin R(\alpha)$, then there exists a local a -stable manifold N around p with respect to f , which is a submanifold of M .

For $a \geq 1$, we have:

- (d) If $a \notin R(\alpha)$, then there exists a local a -unstable manifold N around p with respect to f , which is a submanifold of M .

In all of (a)–(d), the germ of N at p (as an analytic manifold) is uniquely determined. Moreover, there is a basis of open neighbourhoods N' of p in N such that N' has the property of N described in (a)–(d), respectively.

If $\alpha := T_p(f): T_p(M) \rightarrow T_p(M)$ is an automorphism in the preceding situation, then properties of the spectrum of α and properties of the fixed point p of f can be related. The next theorem collects results of this type from Propositions 3.2, 3.3 and 3.5. We say that a fixed point $p \in M_0$ of $f: M_0 \rightarrow M$ is *uniformly attractive* if each neighbourhood of p in M_0 contains a neighbourhood Q of p in M_0 such that $f(Q) \subseteq Q$ and

$$\lim_{n \rightarrow \infty} f^n(x) = p \quad \text{for all } x \in Q$$

(cf. Definition 3.4).

Theorem B. *Let M be a finite-dimensional analytic manifold over a complete ultrametric field \mathbb{K} , $M_0 \subseteq M$ be an open subset, $f: M_0 \rightarrow M$ be an analytic map and $p \in M_0$ a fixed point of f such that $\alpha := T_p(f)$ is an automorphism. Then (a)–(c) hold:*

- (a) $R(\alpha) \subseteq]0, 1]$ if and only if each neighbourhood P of p in M_0 contains a neighbourhood Q of p such that $f(Q) \subseteq Q$;
- (b) $R(\alpha) \subseteq \{1\}$ if and only if each neighbourhood P of p in M_0 contains a neighbourhood Q of p such that $f(Q) = Q$;
- (c) $R(\alpha) \subseteq]0, 1[$ if and only if p is a uniformly attractive fixed point of f .

In the 1-dimensional case, fixed (and periodic) points were already classified into attractive, repelling and indifferent ones in [15]. Results concerning attractive and repelling fixed points, as well as Siegel disks were also obtained in [1], which amount to the sufficiency (but not the necessity) of the spectral condition in (b) and (c) of Theorem B.

It is useful to have conditions ensuring that the (global) stable manifold W^s is not only an immersed submanifold, but a submanifold. In view of Theorem A, our Proposition 4.1 below subsumes the following:

Theorem C. *Let M be a finite-dimensional analytic manifold over a complete ultrametric field. Let $p \in M$ be a fixed point of an analytic diffeomorphism $f: M \rightarrow M$, and $\alpha := T_p(f)$. If $R(\alpha) \subseteq]0, 1]$, then $W_a^s(f, p)$ is a submanifold of M , for each $a \in]0, 1]$ such that $T_p(f)$ is a -hyperbolic.*

If $\beta: G \rightarrow G$ is an automorphism of a finite-dimensional analytic Lie group G over a complete ultrametric field, then the neutral element $e \in G$ is a fixed point for β , but we cannot expect in general that $T_e(\beta)$ is hyperbolic. Nonetheless, it is always possible to turn the stable set

$$U_\beta := W^s(\beta, e) := \{x \in G: \lim_{n \rightarrow \infty} \beta^n(x) = e\}$$

(the so-called *contraction group*) into a manifold:

Theorem D. *If $\beta: G \rightarrow G$ is an automorphism of a finite-dimensional analytic Lie group G over a complete ultrametric field, then there is a unique immersed submanifold structure on $U_\beta = W^s(\beta, e)$ such that conditions (a)–(c) of the Ultrametric Stable Manifold Theorem (with β in place of f) are satisfied. This immersed submanifold structure makes U_β an immersed Lie subgroup of G .*

To explain the motivation for the current article, and to show the utility of its results, we now briefly describe three Lie-theoretic applications which are only available through the use of invariant manifolds.

Applications in Lie theory. Let G be an analytic finite-dimensional Lie group over a local field \mathbb{K} and $\beta: G \rightarrow G$ be an analytic automorphism. The *Levi factor* of β is the subgroup

$$M_\beta := \{x \in G: \beta^{\mathbb{Z}}(x) \text{ is relatively compact in } G\},$$

where $\beta^{\mathbb{Z}}(x) := \{\beta^n(x): n \in \mathbb{Z}\}$ (see [2]). Using invariant manifolds, one can prove the following results in arbitrary characteristic (the p -adic case of which is due to J. S. P. Wang [21]):

- (a) *The group U_β is always nilpotent* (see [7, Theorem B]).

- (b) If U_β is closed, then U_β , $U_{\beta^{-1}}$ and M_β are Lie subgroups of G . Moreover, $U_\beta M_\beta U_{\beta^{-1}}$ is an open subset of G and the “product map”

$$\pi: U_\beta \times M_\beta \times U_{\beta^{-1}} \rightarrow U_\beta M_\beta U_{\beta^{-1}}, \quad (x, y, z) \mapsto xyz$$

is an analytic diffeomorphism (see [10]).

In fact, the a_j -stable manifolds $G_j := W_{a_j}^s(\beta, e)$ provide a central series $\{1\} = G_1 \subseteq G_2 \subseteq \dots \subseteq G_n = G$ of Lie subgroups of G , for suitable real numbers $0 < a_1 < \dots < a_n < 1$ (see [7]). And to get (b), one heavily uses the (stable) manifold structures on $U_\beta = W^s(\beta, e)$ and $U_{\beta^{-1}} = W^s(\beta^{-1}, e)$ discussed here, and the fact that M_β contains a centre manifold for β around e (see [10]; the result was also announced with a sketch of proof in [8, Theorem 9.1]).

- (c) Using (b) as a tool, it is also possible to calculate the “scale” $s(\beta)$ (introduced in [24], [25])² if U_β is closed, in terms of the eigenvalues of the tangent map $L(\beta) := T_e(\beta)$ (see [10]; cf. [8, Theorem 9.3] for a more detailed announcement with a sketch of proof). Previously, this was only possible in the p -adic case (see [5]; cf. also [2] for the scale of inner automorphisms of reductive algebraic groups).

Structure of the article. We first provide notation, basic facts and further definitions of invariant vector subspaces in a preparatory section (Section 1). Sections 2–6 are devoted to the proofs of Theorems A–D, and related results.

1 Preliminaries and notation

In this section, we fix notation and recall some basic facts. We also define (and briefly discuss) centre subspaces and centre-stable subspaces.

In this article, $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We write \mathbb{Z} for the integers and \mathbb{R} for the field of real numbers. If $f: M \rightarrow M$ and $n \in \mathbb{N}$, we write $f^n := f \circ \dots \circ f$ for the n -fold composition, and $f^0 := \text{id}_M$. If f is invertible, we define $f^{-n} := (f^{-1})^n$.

Recall that an *ultrametric field* is a field \mathbb{K} , together with an absolute value $|\cdot|: \mathbb{K} \rightarrow [0, \infty[$ which satisfies the ultrametric inequality. We shall always

²The scale can be defined as the minimum index $s(\beta) := \min_V [V : V \cap \beta^{-1}(V)]$, for V ranging through the set of all compact, open subgroups of G .

assume that the metric $d: \mathbb{K} \times \mathbb{K} \rightarrow [0, \infty[$, $d(x, y) := |x - y|$, defines a non-discrete topology on \mathbb{K} . If the metric space (\mathbb{K}, d) is complete, then the ultrametric field (\mathbb{K}, d) is called *complete*. A totally disconnected, locally compact, non-discrete topological field is called a *local field*. Any such admits an ultrametric absolute value making it a complete ultrametric field [22]. See, e.g., [17] for background concerning complete ultrametric fields.

An *ultrametric Banach space* over an ultrametric field \mathbb{K} is a complete normed space $(E, \|\cdot\|)$ over \mathbb{K} whose norm $\|\cdot\|: E \rightarrow [0, \infty[$ satisfies the *ultrametric inequality*, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in E$ (cf. [20]). The ultrametric inequality entails that

$$\|x + y\| = \|x\| \quad \text{for all } x, y \in E \text{ such that } \|y\| < \|x\|. \quad (2)$$

Given $x \in E$ and $r \in]0, \infty]$, we set $B_r^E(x) := \{y \in E: \|y - x\| < r\}$.

If $A: E \rightarrow F$ is a continuous linear map between ultrametric Banach spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, we write $\|A\| := \sup\{\|Ax\|_F / \|x\|_E: 0 \neq x \in E\}$ for its operator norm. The following observation is immediate.

1.1 If $(E, \|\cdot\|)$ is an ultrametric Banach space over \mathbb{K} and $A: E \rightarrow E$ an invertible continuous linear map, then $\frac{1}{\|A^{-1}\|}$ can be interpreted as an expansion factor, in the sense that $\|Ay\| \geq \frac{1}{\|A^{-1}\|} \|y\|$ for all $y \in E$ (as in the familiar case of real Banach spaces).

We refer to [3] for the concept of an analytic map $f: U \rightarrow F$, where $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are ultrametric Banach spaces and U is an open subset of E ; compare [18] if E and F have finite dimension. Thus, in the terminology of Non-Archimedean Geometry, the mappings we consider are *locally* analytic maps. If f is as before and $x \in U$, we write $f'(x): E \rightarrow F$ for the total differential of f at x . We shall use that f is *strictly* differentiable at x (see [3]):

1.2 If $f: E \supseteq U \rightarrow F$ is analytic and $x \in U$, write

$$f(y) = f(x) + f'(x).(y - x) + R(y) \quad \text{for } y \in U. \quad (3)$$

Then $R|_{B_\varepsilon^E(x)}$ is Lipschitz for small $\varepsilon > 0$ in the sense that

$$\text{Lip}(R|_{B_\varepsilon^E(x)}) := \sup \left\{ \frac{\|R(z) - R(y)\|_F}{\|z - y\|_E} : y \neq z \in B_\varepsilon^E(x) \right\} < \infty,$$

and

$$\lim_{\varepsilon \rightarrow 0} \text{Lip}(R|_{B_\varepsilon^E(x)}) = 0.$$

If $E = F$ and $f'(x)$ is an automorphism, then

$$\text{Lip}(R|_{B_\varepsilon^E(x)}) < \frac{1}{\|f'(x)^{-1}\|}$$

for $\varepsilon > 0$ small enough. Hence, by (2) and (3), for all $y, z \in B_\varepsilon^E(x)$ we have

$$\|f(z) - f(y)\| = \|f'(x)(z - y) + R(z) - R(y)\| = \|f'(x).(z - y)\|. \quad (4)$$

An *analytic manifold* modelled on an ultrametric Banach space E over a complete ultrametric field \mathbb{K} is defined as usual (as a Hausdorff topological space M , together with a (maximal) set \mathcal{A} of homeomorphisms (“charts”) $\phi: U_\phi \rightarrow V_\phi$ from open subsets of M onto open subsets of E , such that $M = \bigcup_{\phi \in \mathcal{A}} U_\phi$ and the mappings $\phi \circ \psi^{-1}$ are analytic for all $\phi, \psi \in \mathcal{A}$). Also the tangent space $T_p M$ of M at $p \in M$, the tangent bundle TM , analytic maps $f: M \rightarrow N$ between analytic manifolds, and the tangent maps $T_p f: T_p M \rightarrow T_{f(p)} N$ as well as $Tf: TM \rightarrow TN$ can be defined as usual (cf. [3]). If $f: M \rightarrow V$ is an analytic map to an open subset V of an ultrametric Banach space F , then we identify TV with $V \times F$ in the natural way and let $df: TM \rightarrow F$ be the second component of the map $Tf: M \rightarrow V \times F$. An *analytic Lie group* G over \mathbb{K} is a group, equipped with an analytic manifold structure modelled on an ultrametric Banach space over \mathbb{K} , such that the group inversion and group multiplication are analytic (cf. [4]). As usual, we write $L(G) := T_e(G)$ and $L(\beta) := T_e(\beta)$, if $\beta: G \rightarrow H$ is an analytic homomorphism between analytic Lie groups. Let M be an analytic manifold modelled on an ultrametric Banach space E . A subset $N \subseteq M$ is called a *submanifold* of M if there exists a complemented vector subspace F of the modelling space of M such that each point $p \in N$ is contained in the domain U of some chart $\phi: U \rightarrow V$ of M such that $\phi(N \cap U) = F \cap V$. By contrast, an analytic manifold N is called an *immersed submanifold* of M if $N \subseteq M$ as a set and the inclusion map $\iota: N \rightarrow M$ is an immersion. Subgroups of Lie groups with analogous properties are called *Lie subgroups* and *immersed Lie subgroups*, respectively. If we call a mapping f an analytic diffeomorphism between two manifolds (or an analytic automorphism of a Lie group), then also the inverse map f^{-1} is assumed analytic.

Let us now complete the definitions of invariant vector subspaces from the

Introduction. In the remainder of this section, let E be an ultrametric Banach space over \mathbb{K} . Let $\alpha: E \rightarrow E$ be a continuous linear map, and $a \in]0, \infty[$.

Remark 1.3 We mention that the spaces $E_{a,s}$ and $E_{a,u}$ in the definition of a -hyperbolicity stated in the Introduction are uniquely determined, in the case of an endomorphism $\alpha: E \rightarrow E$ of a finite-dimensional \mathbb{K} -vector space E . See [9, Remark 6.4] for the assertion if α is an automorphism. In the general case, the argument in the cited remark still provides uniqueness of $E_{a,s}$. Let us write $E^+ := \bigcap_{k \in \mathbb{N}} \alpha^k(E)$ for the Fitting one component of E (see, e.g., [12, Lemma 5.3.11]). Then α restricts to an automorphism β of E^+ . Now $E^+ = (E_{a,s})^+ \oplus E_{a,u}$ is a decomposition for the a -hyperbolic automorphism β and thus also $E_{a,u}$ is unique.

Definition 1.4 An α -invariant vector subspace $E_{a,cs} \subseteq E$ is called an a -centre-stable subspace with respect to α if there exists an α -invariant vector subspace $E_{a,u}$ of E such that $E = E_{a,cs} \oplus E_{a,u}$ and $\alpha_2 := \alpha|_{E_{a,u}}: E_{a,u} \rightarrow E_{a,u}$ is invertible, and there exists an ultrametric norm $\|\cdot\|$ on E defining its topology, with the following properties:

- (a) $\|x + y\| = \max\{\|x\|, \|y\|\}$ for all $x \in E_{a,cs}$, $y \in E_{a,u}$; and
- (b) $\|\alpha_1\| \leq a$ and $\frac{1}{\|\alpha_2^{-1}\|} > a$ holds for the operator norms with respect to $\|\cdot\|$, where $\alpha_1 := \alpha|_{E_{a,cs}}$.

Then $E_{a,cs}$ is uniquely determined and if α is invertible, then $E_{a,u}$ is unique (see [9, Remark 3.3]). Arguing as in Remark 1.3, we see that $E_{a,u}$ is also unique if E is finite-dimensional.

Definition 1.5 We say that an α -invariant vector subspace $E_{a,c} \subseteq E$ is an a -centre subspace with respect to α if there exist α -invariant vector subspaces $E_{a,s}$ and $E_{a,u}$ of E such that $E = E_{a,s} \oplus E_{a,c} \oplus E_{a,u}$, and an ultrametric norm $\|\cdot\|$ on E defining its topology, with the following properties:

- (a) $\|x + y + z\| = \max\{\|x\|, \|y\|, \|z\|\}$ for all $x \in E_{a,s}$, $y \in E_{a,c}$ and $z \in E_{a,u}$;
- (b) $\|\alpha(x)\| = a\|x\|$ for all $x \in E_{a,c}$;
- (c) $\alpha_3 := \alpha|_{E_{a,u}}$ is invertible,³ and

³This hypothesis can be omitted (as it then follows from the others) if E has finite dimension (since $\ker \alpha \subseteq E_{a,s}$) or α is an automorphism.

(d) $\|\alpha_1\| < a$ and $\frac{1}{\|\alpha_3^{-1}\|} > a$ hold for the operator norms with respect to $\|\cdot\|$, where $\alpha_1 := \alpha|_{E_{a,s}}$.

If α is an automorphism, then $E_{a,s}$, $E_{a,c}$ and $E_{a,u}$ are uniquely determined (see [9, Remark 4.3]). If E is finite-dimensional, then $E_{a,s}$ is unique by its description in [9, Remark 4.3], and hence also $E_{a,c}$ and $E_{a,u}$ are unique by the argument from Remark 1.3. $E_{a,s}$ and $E_{a,u}$ are called the *a-stable* and *a-unstable* subspaces of E with respect to α , respectively. If $a = 1$, we simply speak of stable, centre and unstable subspaces, and write E_s , E_c and E_u instead of $E_{1,s}$, $E_{1,c}$ and $E_{1,u}$.

2 Spectral interpretation of hyperbolicity

In this section, we consider the special case where α is an automorphism of a *finite-dimensional* vector space over a complete ultrametric field $(\mathbb{K}, |\cdot|)$. We shall interpret *a-hyperbolicity* as the absence of eigenvalues of absolute value a (in an algebraic closure of \mathbb{K}). Moreover, we shall see that an *a-centre* subspace and an *a-centre-stable* subspace always exist.

2.1 Let $(\mathbb{K}, |\cdot|)$ be a complete ultrametric field, E be a finite-dimensional \mathbb{K} -vector space, and $\alpha: E \rightarrow E$ be a linear map. We define $\overline{\mathbb{K}}$, the extension $|\cdot|$ and $R(\alpha)$ as in the Introduction, using the $\overline{\mathbb{K}}$ -linear self-map $\alpha_{\overline{\mathbb{K}}} := \alpha \otimes \text{id}_{\overline{\mathbb{K}}}$ of the $\overline{\mathbb{K}}$ -vector space $E_{\overline{\mathbb{K}}} := E \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ obtained from E by extension of scalars. For each $\lambda \in \overline{\mathbb{K}}$, we let

$$(E_{\overline{\mathbb{K}}})_{(\lambda)} := \{x \in E_{\overline{\mathbb{K}}} : (\alpha_{\overline{\mathbb{K}}} - \lambda)^d x = 0\}$$

be the generalized eigenspace of $\alpha_{\overline{\mathbb{K}}}$ in $E_{\overline{\mathbb{K}}}$ corresponding to λ (where d is the dimension of the \mathbb{K} -vector space E). Given $\rho \in [0, \infty[$, we define

$$(E_{\overline{\mathbb{K}}})_{\rho} := \bigoplus_{|\lambda|=\rho} (E_{\overline{\mathbb{K}}})_{(\lambda)} \subseteq E_{\overline{\mathbb{K}}}, \quad (5)$$

where the sum is taken over all $\lambda \in \overline{\mathbb{K}}$ such that $|\lambda| = \rho$. As usual, we identify E with $E \otimes 1 \subseteq E_{\overline{\mathbb{K}}}$.

The following fact (cf. (1.0) on p. 81 in [16, Chapter II]) is important:⁴

⁴In [16, p. 81], \mathbb{K} is a local field, but the proof works also for complete ultrametric fields.

Lemma 2.2 *For each $\rho \in R(\alpha)$, the vector subspace $(E_{\overline{\mathbb{K}}})_\rho$ of $E_{\overline{\mathbb{K}}}$ is defined over \mathbb{K} , i.e., $(E_{\overline{\mathbb{K}}})_\rho = (E_\rho)_{\overline{\mathbb{K}}}$ with $E_\rho := (E_{\overline{\mathbb{K}}})_\rho \cap E$. Thus*

$$E = \bigoplus_{\rho \in R(\alpha)} E_\rho, \quad (6)$$

and each E_ρ is an α -invariant vector subspace of E . \square

It is essential for us that certain well-behaved norms exist on E (as in 2.1).

Definition 2.3 A norm $\|\cdot\|$ on E is *adapted to α* if the following holds:

- (a) $\|\cdot\|$ is ultrametric;
- (b) $\left\| \sum_{\rho \in R(\alpha)} x_\rho \right\| = \max\{\|x_\rho\| : \rho \in R(\alpha)\}$ for each $(x_\rho)_{\rho \in R(\alpha)} \in \prod_{\rho \in R(\alpha)} E_\rho$;
and
- (c) $\|\alpha(x)\| = \rho\|x\|$ for each $0 \neq \rho \in R(\alpha)$ and $x \in E_\rho$.

Proposition 2.4 *Let E be a finite-dimensional vector space over a complete ultrametric field $(\mathbb{K}, |\cdot|)$ and $\alpha: E \rightarrow E$ be a linear map. Let $\varepsilon > 0$ and $E_0 := \{x \in E : (\exists n \in \mathbb{N}) \alpha^n(x) = 0\}$. Then E admits a norm $\|\cdot\|$ adapted to α , such that $\alpha|_{E_0}$ has operator norm $< \varepsilon$ with respect to $\|\cdot\|$.*

The proof uses the following lemma:

Lemma 2.5 *For each $\rho \in R(\alpha) \setminus \{0\}$, there exists an ultrametric norm $\|\cdot\|_\rho$ on E_ρ such that $\|\alpha(x)\|_\rho = \rho\|x\|_\rho$ for each $x \in E_\rho$. \square*

Proof. If α is an automorphism, then the assertion holds by [8, Lemma 4.4]. The general case follows if we replace α by the map $\alpha|_{E_\rho}: E_\rho \rightarrow E_\rho$, which is an automorphism as $\ker(\alpha) \subseteq E_0$ and thus $E_\rho \cap \ker(\alpha) = \{0\}$. \square

The next lemma takes care of the case $\rho = 0$.

Lemma 2.6 *Let E be a finite-dimensional vector space over a complete ultrametric field $(\mathbb{K}, |\cdot|)$ and $\alpha: E \rightarrow E$ be a nilpotent linear map. Let $\varepsilon > 0$. Then there exists an ultrametric norm $\|\cdot\|$ on E with respect to which α has operator norm $< \varepsilon$.*

Proof. Assume first that there exists a basis v_1, \dots, v_m of E with respect to which α has Jordan normal form with a single Jordan block, i.e., $\alpha(v_1) = 0$ and $\alpha(v_k) = v_{k-1}$ for $k \in \{2, \dots, m\}$. The case $E = \{0\}$ being trivial, we may assume that $m \geq 1$. Choose $\lambda \in \mathbb{K}$ such that $0 < |\lambda| < \varepsilon$ and define $w_k := \lambda^k v_k$ for $k \in \{1, \dots, m\}$. Then $\alpha(w_k) = \lambda^k v_{k-1} = \lambda w_{k-1}$ for $k \in \{2, \dots, m\}$ and $\alpha(w_1) = 0$, entailing that α has operator norm $< \varepsilon$ with respect to the maximum norm $\|\cdot\|$ on E with respect to the basis w_1, \dots, w_m ,

$$\left\| \sum_{k=1}^m t_k w_k \right\| := \max\{|t_k| : k = 1, \dots, m\} \quad \text{for } t_1, \dots, t_m \in \mathbb{K}.$$

In the general case, we write E as a direct sum $\bigoplus_{j=1}^n E_j$ of α -invariant vector subspaces $E_j \subseteq E$ such that the Jordan decomposition of $\alpha|_{E_j}$ has a single Jordan block. For each j , there exists an ultrametric norm $\|\cdot\|_j$ on E_j with respect to which $\alpha|_{E_j}$ has operator norm $< \varepsilon$, by the above special case. Then α has operator norm $< \varepsilon$ with respect to the ultrametric norm $\|\cdot\|$ on E given by $\|v_1 + \dots + v_n\| := \max\{\|v_j\|_j : j = 1, \dots, n\}$ for $v_j \in E_j$. \square

Proof of Proposition 2.4. For each $\rho \in R(\alpha) \setminus \{0\}$, we choose a norm $\|\cdot\|_\rho$ on E_ρ as described in Lemma 2.5. Lemma 2.6 provides an ultrametric norm $\|\cdot\|_0$ on E_0 , with respect to which $\alpha|_{E_0}$ has operator norm $< \varepsilon$. Then

$$\left\| \sum_{\rho \in R(\alpha)} x_\rho \right\| := \max \left\{ \|x_\rho\|_\rho : \rho \in R(\alpha) \right\} \quad \text{for } (x_\rho)_{\rho \in R(\alpha)} \in \prod_{\rho \in R(\alpha)} E_\rho$$

defines a norm $\|\cdot\| : E \rightarrow [0, \infty[$ which, by construction, is adapted to α and with respect to which $\alpha|_{E_0}$ has operator norm $< \varepsilon$. \square

We are now ready to prove Theorem A from the Introduction.

Proof of Theorem A. By Proposition 2.4, there exists an ultrametric norm $\|\cdot\|^\sim$ on E which is adapted to α , and with respect to which $\alpha|_{E_0}$ has operator norm $< a$.

Centre-stable subspaces. The conditions from Definition 1.4 are satisfied with $\|\cdot\| := \|\cdot\|^\sim$ and

$$E_{a,cs} := \bigoplus_{\rho \leq a} E_\rho \quad \text{and} \quad E_{a,u} := \bigoplus_{\rho > a} E_\rho. \quad (7)$$

Centre subspaces. The conditions of Definition 1.5 are satisfied with $\|\cdot\| := \|\cdot\|^\sim$ and

$$E_{a,s} := \bigoplus_{\rho < a} E_\rho, \quad E_{a,c} := E_a, \quad \text{and} \quad E_{a,u} := \bigoplus_{\rho > a} E_\rho. \quad (8)$$

Hyperbolicity. If $a \notin R(\alpha)$, then the conditions from the definition of a -hyperbolicity (stated in the Introduction) are satisfied with $\|\cdot\| := \|\cdot\|^\sim$,

$$E_{a,s} := \bigoplus_{\rho < a} E_\rho \quad \text{and} \quad E_{a,u} := \bigoplus_{\rho > a} E_\rho. \quad (9)$$

If $a \in R(\alpha)$, then α cannot be a -hyperbolic. In fact, if α was a -hyperbolic, we obtain a norm $\|\cdot\|$ and a splitting $E = E_{a,s} \oplus E_{a,u}$ as in the cited definition. Define $\alpha_1 := \alpha|_{E_{a,s}}$ and $\alpha_2 := \alpha|_{E_{a,u}}$. Because the norms $\|\cdot\|$ and $\|\cdot\|^\sim$ are equivalent, there exists $C > 0$ such that $C^{-1}\|\cdot\| \leq \|\cdot\|^\sim \leq C\|\cdot\|$. Let $0 \neq v \in E_a$. Write $v = x + y$ with $x \in E_{a,s}$ and $y \in E_{a,u}$. If $y \neq 0$, then

$$\|v\|^\sim = a^{-n} \|\alpha^n(v)\|^\sim \geq a^{-n} C^{-1} \|\alpha^n(v)\| \geq C^{-1} \left(\frac{1}{a \|\alpha_2^{-1}\|} \right)^n \|y\|$$

for all $n \in \mathbb{N}$, which is absurd because $\frac{1}{a \|\alpha_2^{-1}\|} > 1$. Hence $y = 0$ and thus $x = v \neq 0$. But then

$$\|v\|^\sim = a^{-n} \|\alpha^n(v)\|^\sim \leq a^{-n} C \|\alpha^n(v)\| \leq C \left(\frac{\|\alpha_1\|}{a} \right)^n \|v\| \quad \text{for all } n \in \mathbb{N}.$$

Since $\frac{\|\alpha_1\|}{a} < 1$, this is absurd. Thus α cannot be a -hyperbolic. \square

3 Behaviour close to a fixed point

We now relate the behaviour of a dynamical system (M, f) around a fixed point p and properties of the linear map $T_p(f)$.

3.1 Let M be an analytic manifold modelled on an ultrametric Banach space over a complete ultrametric field $(\mathbb{K}, |\cdot|)$. Let $f: M_0 \rightarrow M$ be an analytic mapping on an open subset $M_0 \subseteq M$ and $p \in M_0$ be a fixed point of f , such that $T_p(f): T_p(M) \rightarrow T_p(M)$ is an automorphism.

Proposition 3.2 *In 3.1, the following conditions are equivalent:*

- (a) $T_p(M)$ admits a centre-stable subspace with respect to $T_p(f)$, and each neighbourhood P of p in M_0 contains a neighbourhood Q of p such that $f(Q) \subseteq Q$.
- (b) There exists a norm $\|\cdot\|$ on $T_p(M)$ defining its topology, such that $\|T_p(f)\| \leq 1$ holds for the corresponding operator norm.

If, moreover, M is a finite-dimensional manifold, then (a) and (b) are also equivalent to the following condition:

- (c) Each eigenvalue λ of $T_p(f) \otimes_{\mathbb{K}} \text{id}_{\overline{\mathbb{K}}}$ in an algebraic closure $\overline{\mathbb{K}}$ of \mathbb{K} has absolute value $|\lambda| \leq 1$.

Proof. (b) means that $E := T_p(M)$ coincides with its centre-stable subspace with respect to $\alpha := T_p(f)$. If E is finite-dimensional, this property is equivalent to $R(\alpha) \subseteq]0, 1]$ and hence to (c), by (7) (using that E_{cs} is unique). If (b) holds, then (a) follows with [9, Theorem 1,9 (c)].⁵

(a) \Rightarrow (b): If (a) holds, then E admits a decomposition $E = E_{1,\text{cs}} \oplus E_{1,\text{u}}$ and a norm $\|\cdot\|$, as described in Definition 1.4 (with $a = 1$). After shrinking M_0 , we may assume that $M_1 := f(M_0)$ is open in M and $f: M_0 \rightarrow M_1$ is a diffeomorphism (by the Inverse Function Theorem).

If $E_{1,\text{u}} \neq \{0\}$, we let $P \subseteq M_0 \cap M_1$ be an open neighbourhood of p such that $f(P) \subseteq P$, and consider the map $g := f^{-1}: M_1 \rightarrow M$. Then $E_{1,\text{u}}$ is the stable subspace of E with respect to $T_p(g) = \alpha^{-1}$. Pick $b \in]\|\alpha^{-1}|_{E_{1,\text{u}}}\|, 1[$. Then α^{-1} is b -hyperbolic, and

$$E_{b,\text{s}} = E_{1,\text{u}} \quad \text{as well as} \quad E_{b,\text{u}} = E_{1,\text{cs}}$$

(with respect to the automorphisms α^{-1} and α on the left and right of the equality signs, respectively). By [9, Theorem 6.6] (applied to $g|_P: P \rightarrow M$), there exists a local b -stable manifold $N \subseteq P$ with respect to g , such that $g^n(x) \rightarrow p$ as $n \rightarrow \infty$, for all $x \in N$. Since N is tangent to $E_{1,\text{u}} \neq \{0\}$, we have $N \neq \{p\}$ and thus find a point $x \in N \setminus \{p\}$. By hypothesis (a), there is an open p -neighbourhood $Q \subseteq P \setminus \{x\}$ with $f(Q) \subseteq Q$. Since $g^n(x) \rightarrow p$, there exists $m \in \mathbb{N}$ with $y := g^m(x) \in Q$. Then $x = f^m(y) \in f^m(Q) \subseteq Q$, contradicting the choice of Q . Hence $E_{1,\text{u}} = \{0\}$ (and thus (b) holds). \square

⁵If E is finite-dimensional, this corresponds to the conclusions concerning centre-stable manifolds in the Local Invariant Manifold Theorem stated above.

Proposition 3.3 *In 3.1, the following conditions are equivalent:*

- (a) $T_p(M)$ admits a centre subspace with respect to $T_p(f)$, and each neighbourhood P of p in M_0 contains a neighbourhood Q of p such that $f(Q) = Q$.
- (b) There exists a norm $\|\cdot\|$ on $T_p(M)$ defining its topology, which makes $T_p(f)$ an isometry.

If, moreover, M is a finite-dimensional manifold, then (a) and (b) are also equivalent to the following condition:

- (c) Each eigenvalue λ of $T_p(f) \otimes_{\mathbb{K}} \text{id}_{\overline{\mathbb{K}}}$ in an algebraic closure $\overline{\mathbb{K}}$ of \mathbb{K} has absolute value $|\lambda| = 1$.

Proof. (b) means that $E := T_p(M)$ coincides with its centre subspace with respect to $\alpha := T_p(f)$. If E is finite-dimensional, this property is equivalent to $R(\alpha) \subseteq \{1\}$ and hence to (c), by (8) (using the uniqueness of E_c). If (b) holds, then (a) follows with [9, Theorem 1.10 (c)].⁶

(a) \Rightarrow (b): After shrinking M_0 , we may assume that $M_1 := f(M_0)$ is open in M and $f: M_0 \rightarrow M_1$ is a diffeomorphism. If (a) holds, then there is a decomposition $E = E_{1,s} \oplus E_{1,c} \oplus E_{1,u}$ and a norm $\|\cdot\|$, as in Definition 1.5 (with $a = 1$). By “(a) \Rightarrow (b)” in Proposition 3.2, we have $E_{1,u} = \{0\}$. Applying Proposition 3.2 to $g := f^{-1}: M_1 \rightarrow M$, we see that also $E_{1,s} = \{0\}$ (because this is the unstable subspace of $T_p(M)$ with respect to $T_p(g) = \alpha^{-1}$). Thus $E = E_{1,c}$, establishing (b). \square

The proofs show that Q can always be chosen as an *open* subset of M_0 , in part (a) of Proposition 3.2 and 3.3.

Definition 3.4 In the situation of 3.1, we use the following terminology:

- (a) p is said to be an *attractive* fixed point of f if p has a neighbourhood $P \subseteq M_0$ such that $f^n(x)$ is defined for all $x \in P$ and $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} f^n(x) = p$ for all $x \in P$.
- (b) We say that p is *uniformly attractive* if it is attractive and, moreover, every neighbourhood of p in M_0 contains a neighbourhood Q of p such that $f(Q) \subseteq Q$.

⁶If E is finite-dimensional, see also the conclusions concerning centre manifolds in the Local Invariant Manifold Theorem above.

Proposition 3.5 *In 3.1, the following conditions are equivalent:*

- (a) $T_p(M)$ admits a centre subspace with respect to $T_p(f)$, and p is uniformly attractive;
- (b) There exists a norm $\|\cdot\|$ on $T_p(M)$ defining its topology, such that $\|T_p(f)\| < 1$ holds for the corresponding operator norm.

If, moreover, M is a finite-dimensional manifold, then (a) and (b) are also equivalent to the following condition:

- (c) Each eigenvalue λ of $T_p(f) \otimes_{\mathbb{K}} \text{id}_{\overline{\mathbb{K}}}$ in an algebraic closure $\overline{\mathbb{K}}$ of \mathbb{K} has absolute value $|\lambda| < 1$.

Proof. (b) means that $E := T_p(M)$ coincides with its stable subspace with respect to $\alpha := T_p(f)$. If E is finite-dimensional, this property is equivalent to $R(\alpha) \subseteq]0, 1[$ and hence to (c), by (8) (using the uniqueness of E_s). If (a) holds, then also (b), as shall be verified in Remark 3.6.

If (b) holds and $P \subseteq M_0$ is an open neighbourhood of p , then [9, Theorem 6.6]⁷ (applied to $f|_P$ instead of f) provides a local stable manifold $N \subseteq P$ such that $\lim_{n \rightarrow \infty} f^n(x) = p$ for all $x \in N$. Because $T_p(N) = E = T_p(M)$, it follows that N is open in M . Since, moreover, $f(N) \subseteq N$ by definition of N , we have verified that p is uniformly attractive. \square

Remark 3.6 If p is merely attractive (but possibly not uniformly) and $E := T_p(M)$ admits a centre subspace with respect to $T_p(f)$, we can still conclude that $E_{1,c} = \{0\}$.

[After shrinking M_0 , we may assume that f is injective. Let $P \subseteq M_0$ be as in Definition 3.4 (a). If $E_{1,c} \neq \{0\}$, we let $Q \subseteq P$ be a centre manifold with respect to f , such that $f(Q) = Q$ (see [9, Theorem 1.10 (c)]). Since $E_{1,c} \neq \{0\}$, we must have $Q \neq \{p\}$, enabling us to pick $x_0 \in Q \setminus \{p\}$. Using [9, Theorem 1.10 (c)] again, we find a centre manifold $S \subseteq Q \setminus \{x_0\}$ with respect to f , such that $f(S) = S$. Since f is injective, it follows that $f(Q \setminus S) = Q \setminus S$ and thus $f^n(x_0) \in Q \setminus S$ for all $n \in \mathbb{N}_0$. As Q is a neighbourhood of p , we infer $f^n(x_0) \not\rightarrow p$ as $n \rightarrow \infty$. Since $x_0 \in P$, this contradicts the choice of P .]

⁷If E is finite-dimensional, see also the conclusions concerning local stable manifolds in the Local Invariant Manifold Theorem above.

4 When $W_a^s(f, p)$ is not only immersed

In general, W_a^s is only an *immersed* submanifold of M , not a submanifold (cf. [8, §7.1] for an easy example). We now describe a criterion (needed in [7]) which prevents such pathologies.

Proposition 4.1 *Let M be an analytic manifold modelled on an ultrametric Banach space over a complete ultrametric field. Let $p \in M$ be a fixed point of an analytic diffeomorphism $f: M \rightarrow M$, such that $E := T_p(M)$ admits a centre-stable subspace with respect to $T_p(f)$, and $E_{1,u} = \{0\}$. Then $W_a^s(f, p)$ is a submanifold of M , for each $a \in]0, 1]$ such that $T_p(f)$ is a -hyperbolic.*

Proof. Let $W_a^s := W_a^s(f, p)$ and $\Omega \subseteq W_a^s$ be as in the Ultrametric Stable Manifold Theorem from the Introduction. Since f restricts to a diffeomorphism of W_a^s , the image $f(\Omega)$ is relatively open in Ω . Hence, there exists an open p -neighbourhood $Q \subseteq M$ such that $\Omega \cap Q \subseteq f(\Omega)$. By “(b) \Rightarrow (a)” in Proposition 3.2, we may assume that $f(Q) \subseteq Q$, after replacing Q with a smaller neighbourhood of p if necessary. We claim that

$$W_a^s \cap Q = \Omega \cap Q. \quad (10)$$

If this is true, then $W_a^s \cap Q$ is a submanifold of M , and hence also

$$f^{-n}(W_a^s \cap Q) = f^{-n}(W_a^s) \cap f^{-n}(Q) = W_a^s \cap f^{-n}(Q)$$

is a submanifold of M (as $f^{-n}: M \rightarrow M$ is a diffeomorphism). Since $\bigcup_{n \in \mathbb{N}_0} f^{-n}(Q)$ is an open subset of M which contains W_a^s (exploiting that $f^n(x) \in Q$ for large n , for each $x \in W_a^s$), we deduce that W_a^s is a submanifold of M (and the submanifold structure coincides with the given immersed submanifold structure on W_a^s , as both structures coincide on each of the sets $f^{-n}(W_a^s \cap Q)$, $n \in \mathbb{N}_0$, which form an open cover for W_a^s).

To prove (10), suppose that $x \in W_a^s \cap Q$ but $x \notin \Omega \cap Q$ (and hence $x \notin \Omega$). Since $f(Q) \subseteq Q$, we then have

$$f^n(x) \in Q \quad \text{for all } n \in \mathbb{N}_0.$$

By definition of Ω , there exists $n \in \mathbb{N}_0$ such that $f^n(x) \in \Omega$. We choose n minimal and note that $n \geq 1$ as $x \notin \Omega$ by hypothesis. Then $f^n(x) \in \Omega \cap Q \subseteq f(\Omega)$ and hence $f^{n-1}(x) = f^{-1}(f^n(x)) \in f^{-1}(f(\Omega)) = \Omega$, contradicting the minimality of n . Hence x cannot exist and thus $W_a^s \cap Q \subseteq \Omega \cap Q$. The converse inclusion, $\Omega \cap Q \subseteq W_a^s \cap Q$, being trivial, (10) is proved. \square

5 Dependence of a -stable manifolds on $a > 0$

We collect further results in the finite-dimensional case required in Section 6 and [7]. In particular, we study the dependence of a -stable manifolds on the parameter a .

Proposition 5.1 *Let M be an analytic manifold modelled on a finite-dimensional vector space over a complete ultrametric field $(\mathbb{K}, |\cdot|)$. Let $p \in M$ be a fixed point of an analytic diffeomorphism $f: M \rightarrow M$. Abbreviate $\alpha := T_p(f)$ and define $R(\alpha)$ as in the Introduction. Then the following holds:*

- (a) *If $R(\alpha) \subseteq]0, 1]$, then $W_a^s(f, p)$ is a submanifold of M , for each $a \in]0, 1] \setminus R(\alpha)$.*
- (b) *If $0 < a < b \leq 1$ and $[a, b] \cap R(\alpha) = \emptyset$, then $W_a^s(f, p) = W_b^s(f, p)$.*
- (c) *If $a \in]0, 1]$ and $]0, a] \cap R(\alpha) = \emptyset$, then $W_a^s(f, p) = \{p\}$.*

Proof. (a) follows from Proposition 4.1 (using (8) and Theorem A).

(b) Define $E := T_p(M)$. Let $\|\cdot\|$ be a norm on E adapted to $\alpha := T_p(f)$, and $R(\alpha)$ as well as the subspaces $E_\rho \subseteq E$ for $\rho > 0$ be as in 2.1. Choose a chart $\kappa: P \rightarrow U \subseteq E$ of M around p such that $\kappa(p) = 0$. Let $Q \subseteq P$ be an open neighbourhood of p such that $f(Q) \subseteq P$; after shrinking Q , we may assume that $\kappa(Q) = B_r^E(0)$ for some $r > 0$. Then $g := \kappa \circ f|_Q \circ \kappa^{-1}|_{B_r^E(0)}: B_r^E(0) \rightarrow E$ expresses $f|_Q$ in the local chart κ . By hypothesis on a and b , we have

$$X := \bigoplus_{\rho < a} E_\rho = \bigoplus_{\rho < b} E_\rho \quad \text{and} \quad Y := \bigoplus_{\rho > a} E_\rho = \bigoplus_{\rho > b} E_\rho.$$

Hence $E_{a,s} = E_{b,s} = X$ and $E_{a,u} = E_{b,u} = Y$, by (9). Now let Ω_a and Ω_b be an Ω as in the Ultrametric Stable Manifold Theorem, applied with a and b , respectively. By [9, Theorem 6.2 (f)] and the proof of Theorem 1.3 in [9], we may assume that $\Omega_a = \kappa^{-1}(\Gamma_a)$ and $\Omega_b = \kappa^{-1}(\Gamma_b)$, where

$$\begin{aligned} \Gamma_a &= \{z \in B_r^E(0) : (\forall n \in \mathbb{N}_0) \ g^n(z) \text{ is defined and } \|g^n(z)\| \leq a^n r\} \text{ and} \\ \Gamma_b &= \{z \in B_t^E(0) : (\forall n \in \mathbb{N}_0) \ g^n(z) \text{ is defined and } \|g^n(z)\| \leq b^n t\} \end{aligned} \quad (11)$$

for certain $r, t > 0$. Moreover, by [9, Theorem 6.2 (e)], we may assume that $r = t$, after replacing both r and t by $\min\{r, t\}$. Then $\Gamma_a \subseteq \Gamma_b$ by (11), and

hence $\Gamma_a = \Gamma_b$ (since both sets are graphs of functions on the same domain, by the cited theorem). Thus $\Omega_a = \Omega_b$, entailing that $W_a^s(f, p) = W_b^s(f, p)$ as a set and also as an immersed submanifold of M (cf. proof of [9, Theorem 1.3]).

(c) By (9), we have $E_{a,s} = \bigoplus_{\rho < a} E_\rho = \{0\}$, whence $\Omega = \kappa^{-1}(\Gamma) = \{p\}$ in [9, Theorem 1.3] and its proof. Thus $W_a^s(f, p) = \bigcup_{n \in \mathbb{N}_0} f^{-n}(\Omega) = \{p\}$. \square

6 Results for automorphisms of Lie groups

Throughout this section, G is an analytic Lie group modelled on an ultrametric Banach space over a complete ultrametric field $(\mathbb{K}, |\cdot|)$, and $\beta: G \rightarrow G$ an analytic automorphism. Then the neutral element $e \in G$ is a fixed point of β , and hence our general theory applies. We now compile some additional conclusions which are specific to automorphisms. Like results of the previous sections, these are needed for the farther-reaching Lie-theoretic applications described in the introduction.

We begin with a corollary to Proposition 3.5. An automorphism $\beta: G \rightarrow G$ is called *contractive* if $\lim_{n \rightarrow \infty} \beta^n(x) = e$ for each $x \in G$.

Corollary 6.1 *If G is finite-dimensional and $\beta: G \rightarrow G$ a contractive automorphism, then every eigenvalue λ of $L(\beta) \otimes_{\mathbb{K}} \text{id}_{\overline{\mathbb{K}}}$ in an algebraic closure $\overline{\mathbb{K}}$ has absolute value $|\lambda| < 1$.*

Proof. G is complete by [6, Proposition 2.1 (a)], and metrizable. Since every identity neighbourhood P in G contains an open subgroup U of G (see, e.g., [6, Proposition 2.1 (a)]), Lemma 1 (a) in [19] provides a β -invariant open subgroup $Q := U_{(0)} \subseteq U \subseteq P$ of G . Hence e is a uniformly contractive fixed point of β , and thus “(a) \Rightarrow (c)” in Proposition 3.5 applies. \square

Proposition 6.2 *If $a \in]0, 1]$ and $L(\beta)$ is a -hyperbolic, the following holds:*

- (a) *The a -stable manifold $W_a^s(\beta, e)$ is an immersed Lie subgroup of G .*
- (b) *If, moreover, $L(G)$ admits a centre subspace with respect to $L(\beta)$ and $L(G)_{1,u} = \{0\}$, then $W_a^s(\beta, e)$ is a Lie subgroup of G .*

Proof. (a) The proof of [7, Proposition 4.6] applies without changes.⁸

(b) is a special case of Proposition 4.1. \square

If G is finite-dimensional, then the extra hypotheses in Proposition 6.2 (b) mean that $R(L(\beta)) \subseteq]0, 1]$ (see Theorem A and (8)).

In the following situation, hyperbolicity is not needed to make W^s a manifold.

Proposition 6.3 *If $\beta: G \rightarrow G$ is an automorphism and $L(G)$ admits a centre subspace with respect to $L(\beta): L(G) \rightarrow L(G)$, then the following holds:*

- (a) *There exist a local stable manifold V_1 and a centre manifold V_0 around e with respect to β , and a local stable manifold V_{-1} around e with respect to β^{-1} , such that $V_1 V_0 V_{-1}$ is open in G and the product map*

$$\pi: V_1 \times V_0 \times V_{-1} \rightarrow V_1 V_0 V_{-1} \quad (x, y, z) \mapsto xyz \quad (12)$$

is an analytic diffeomorphism.

- (b) *There is a unique immersed submanifold structure on $W^s(\beta, e)$ such that conditions (a)–(c) of the Ultrametric Stable Manifold Theorem (from the Introduction) are satisfied. This immersed submanifold structure makes $W^s(\beta, e)$ an immersed Lie subgroup of G , and also the final assertion of the cited theorem holds. Moreover, $W^s(\beta, e) = W_a^s(\beta, e)$ for some $a \in]0, 1[$ such that $L(\beta)$ is a -hyperbolic.*

Proof. (a) Set $E := L(G)$ and let $E = E_1 \oplus E_0 \oplus E_{-1}$ be the decomposition into a stable subspace E_1 , centre subspace E_0 and unstable subspace E_{-1} with respect to $L(\beta)$, and $\|\cdot\|$ be an ultrametric norm as in Definition 1.5. There is $a \in]0, 1[$ such that $\|L(\beta)|_{E_1}\| < a$ and $\frac{1}{\|L(\beta)^{-1}|_{E_{-1}}\|} > \frac{1}{a}$. Then $L(\beta)$ is a -hyperbolic with a -stable subspace E_1 and a -unstable subspace $E_0 \oplus E_{-1}$ (and the norm $\|\cdot\|$ as before). Also $L(\beta)^{-1}$ is a -hyperbolic, with a -stable subspace E_{-1} and a -unstable subspace $E_0 \oplus E_1$ (and the norm $\|\cdot\|$ as before). We let V_1 be a local a -stable manifold around e with respect to β and V_{-1} be a local a -stable manifold around e with respect to β^{-1} (see [9, Theorem 6.6 (a)]); by [9, Theorem 6.6 (c)], we may assume that $V_1 \subseteq W_a^s(\beta, e)$. Also, we let V_0

⁸In \diamond , read “ $\leq a^n$ ” as “ $< a^n r$.”

be a centre manifold around p with respect to β (see [9, Theorem 1.10 (a)]). Then $T_e(V_1) = E_1$, $T_e(V_0) = E_0$ and $T_e(V_{-1}) = E_{-1}$, whence

$$L(G) = T_e(V_1) \oplus T_e(V_0) \oplus T_e(V_{-1}).$$

Thus, after shrinking V_1 , V_0 and V_{-1} (which is possible by [9, Theorems 6.6 (c) and 1.10 (c)]), we may assume that $P := V_1 V_0 V_{-1}$ is open in G and the product map (12) is an analytic diffeomorphism (by the Inverse Function Theorem [3]).

(b) Shrinking V_1 , V_0 and V_{-1} further if necessary, we may assume that there are $r > 0$ and charts $\kappa_j: V_j \rightarrow B_r^{E_j}(0)$ with $\kappa_j(e) = 0$ and $d\kappa_j = \text{id}$ for $j \in \{-1, 0, 1\}$. There is $s \in]0, r]$ such that $\beta(\kappa_j^{-1}(B_s^{E_j}(0))) \subseteq V_j$ for all $j \in \{-1, 0, 1\}$. Let $g_j := \kappa_j \circ \beta \circ \kappa_j^{-1}|_{B_s^{E_j}(0)}$. Shrinking s , we achieve that

$$\|g_0(x)\| = \|x\| \quad \text{for each } x \in B_s^{E_0}(0), \quad (13)$$

$$\|g_1(x)\| < a\|x\| \quad \text{for each } x \in B_s^{E_1}(0), \text{ and} \quad (14)$$

$$\|g_{-1}(x)\| > a^{-1}\|x\| \quad \text{for each } x \in B_s^{E_{-1}}(0) \quad (15)$$

(using (4)). Then

$$\kappa := (\kappa_1 \times \kappa_0 \times \kappa_{-1}) \circ \pi^{-1}: P \rightarrow B_r^E(0)$$

is a chart of G around e . We set $g := g_1 \times g_0 \times g_{-1}: B_s^E(0) \rightarrow B_r^E(0)$ (where $B_s^E(0) = B_s^{E_1}(0) \times B_s^{E_0}(0) \times B_s^{E_{-1}}(0)$). Abbreviate $Q := \kappa^{-1}(B_s^E(0))$. Then

$$\beta|_Q = \kappa^{-1} \circ g \circ \kappa|_Q. \quad (16)$$

If $z \in W^s(\beta, e)$, there is $n_0 \in \mathbb{N}_0$ such that $\beta^n(z) \in Q$ for all $n \geq n_0$, and

$$\|\kappa(\beta^n(z))\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (17)$$

After replacing z with $\beta^{n_0}(z)$, we may assume that $n_0 = 0$. Now $x = (x_1, x_0, x_{-1}) := \kappa(z)$ is an element of $B_s^E(0)$ such that $g^n(x) = \kappa(\beta^n(z)) \in B_s^E(0)$ for all $n \in \mathbb{N}_0$ (cf. (16)). Also

$$\lim_{n \rightarrow \infty} \|g^n(x)\| = 0, \quad (18)$$

by (17). Since $\|g^n(x)\| = \max\{\|g_1^n(x_1)\|, \|g_0^n(x_0)\|, \|g_{-1}^n(x_{-1})\|\}$ for all $n \in \mathbb{N}_0$, using (13) and (15) we obtain a contradiction to (18) unless $x_0 = 0$

and $x_{-1} = 0$. Thus $x = x_1 \in E_1$ and thus $z = \kappa_1^{-1}(x_1) \in V_1 \subseteq W_a^s(\beta, e)$, entailing that $W^s(\beta, e) \subseteq W_a^s(\beta, e)$. The converse inclusion being trivial, we deduce that $W^s(\beta, e) = W_a^s(\beta, e)$. We give $W^s(\beta, e)$ the manifold structure of $W_a^s(\beta, e)$. It then is tangent to $E_{a,s} = E_1$ at e . Hence $W^s(\beta, e)$ satisfies conditions (a)–(c) of the Ultrametric Stable Manifold Theorem and also the final assertion of the theorem. To obtain the uniqueness of the immersed submanifold structure subject to these conditions, note that for any such structure on W^s , each neighbourhood of e in W^s contains an open β -invariant neighbourhood of e (as this only requires (2) and 1.2). Now one shows as in the proof of [9, Theorem 6.6 (b)] that the germ of the latter coincides with the germ we already have, and this entails as in the proof of the uniqueness part of [9, Theorem 1.3] that the new manifold structure on W^s coincides with the one we already had (further explanations are omitted, because the assertion is not central). All other assertions follow from Proposition 6.2. \square

Proof of Theorem D. We now prove Theorem D. The proof will provide additional information: $W^s(\beta, e) = W_a^s(\beta, e)$ for each $a \in]0, 1[$ such that $[a, 1[\cap R(L(\beta)) = \emptyset$ and $]1, \frac{1}{a}] \cap R(L(\beta)) = \emptyset$.

If we choose $\|\cdot\|$ as a norm adapted to $L(\beta)$ (as in Definition 2.3) in the proof of Proposition 6.3, then E_1 , E_0 and E_{-1} are the direct sum of all $L(G)_\rho$ with $\rho \in R(L(\beta))$, such that $\rho \in]0, 1[$ (resp., $\rho = 1$, resp., $\rho \in]1, \infty[$), by (8). If a is as described at the beginning of the proof, then $\|L(\beta)\| < a$ and $\|L(\beta)^{-1}\| < a$ (as is clear from (b) and (c) in Definition 2.3). Therefore the proof of Proposition 6.3 applies with this choice of a . \square

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